Stochastic Calculus and Anticommuting Variables

Alice Rogers*
Department of Mathematics
King's College
Strand
London WC2R 2LS

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Abstract

A theory of integration for anticommuting paths is described. This is combined with standard Itô calculus to give a geometric theory of Brownian paths on curved supermanifolds.

This lecture concerns a generalisation of Brownian motion and Itô calculus to include paths in spaces of anticommuting variables. The motivation for this work comes originally from physics, where anticommuting variables were first introduced by Martin [7] in order to extend Feynman's path integral methods to Fermionic systems. Subsequently various geometric applications

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to this approach have emerged, and additionally anticommutiong variables have been found to play a significant rôle in the quantizaton of systems with gauge symmetry.

1 Functions of anticommuting variables

Suppose that $\theta^1, \ldots, \theta^n$ are n anticommuting variables, so that

$$\theta^i\theta^j = -\theta^j\theta^i. \tag{1}$$

Then, since $(\theta^i)^2$ is zero, the natural space of functions to consider is the 2^n-dimensional space of functions of the form

$$f(\theta^1, \ldots, \theta^n) = \sum_{\mu\in M_n} f_\mu \theta^\mu \tag{2}$$

where μ denotes a multi-index $\mu_1 \ldots \mu_{|\mu|}$, with $1 \leq \mu_1 < \ldots < \mu_{|\mu|} \leq n$, M_n denotes the set of all such multi-indices (including the empty multi-index $\emptyset$) and $\theta^\mu = \mathbb{1}\theta^{\mu_1} \ldots \theta^{\mu_{|\mu|}}$. The detailed nature of this function space is determined by the choice of space in which the coefficient functions f_μ lie. This may be the real numbers, the complex numbers or some space of commuting and anticommuting variables.

The importance of such functions for Fermionic physics is that they allow realisations of the canonical anticommutation relations

$$\psi^i\psi^j + \psi^j\psi^i = 2\delta^{ij} \tag{3}$$

for Fermionic operators, for instance by setting

$$\psi^i = \theta^i + \frac{\partial}{\partial\theta^i}. \tag{4}$$

The partial derivative in this expression is defined by setting

$$\begin{aligned} \frac{\partial\theta^\mu}{\partial\theta^j} &= (-1)^{\ell-1}\theta^{\mu_1} \ldots \theta^{\mu_{\ell-1}}\theta^{\mu_{\ell+1}} \ldots \theta^{\mu_{|\mu|}} \text{ if } j = \mu_\ell \text{ for some } \ell, 1 \leq \ell \leq |\mu|, \\ \frac{\partial\theta^\mu}{\partial\theta^j} &= 0 \ \text{ otherwise.} \end{aligned} \tag{5}$$

Following Berezin [2], integration is defined by setting

$$\int d^n\theta \, f(\theta^1, \ldots, \theta^n) = f_{1\ldots n} \tag{6}$$

if

$$f(\theta^1, \ldots, \theta^n) = f_{1\ldots n}\theta^1 \ldots \theta^n + \text{ lower order terms.} \tag{7}$$

This integral is formal, but has many useful properties. For instance, suppose that the Fourier transform $\hat{f}$ of the function f is defined by

$$\hat{f}(\rho) = \int_{\mathcal{B}} d^n\theta \, \exp(-i\rho.\theta) f(\theta), \tag{8}$$

(with $\rho.\theta = \sum_{j=1}^{n} \rho^j \theta^j$); then a direct analogue of the Fourier inversion formula can be established by explicit calculation. (Here, and in the following, it will be assumed for simplicity that n is even, because in this case many factors of i can be avoided in the formulae.)

If $\mathcal{O}$ is a linear operator on the space of functions defined above, its kernel is the function $\mathcal{O}(\theta^1, \ldots, \theta^n, \phi^1, \ldots, \phi^n)$ of $2n$ anticommuting variables which satisfies

$$\mathcal{O} f(\theta^1, \ldots, \theta^n) = \int d^n\phi \, \mathcal{O}(\theta^1, \ldots, \theta^n, \phi^1, \ldots, \phi^n) f(\phi^1, \ldots, \phi^n). \tag{9}$$

A particular example of such a kernel is the delta function (that is, the kernel of the identity operator) which takes the form

$$\begin{aligned} \delta(\theta - \phi) &= \int_{\mathcal{B}} d^n\rho \exp(-i\rho.(\theta - \phi)) \\ &= \Pi_{j=1}^{n} (\theta^j - \phi^j). \end{aligned} \tag{10}$$

In most application both anticommuting and commuting variables are required, so that one works on what has become known as a superspace. The notation $\mathbb{R}_S^{m,n}$ will be used for a superspace with m commuting coordinates and n anticommuting coordinates, with a typical point denoted $(x^1, \ldots, x^m; \theta^1, \ldots, \theta^n)$.

2 Fermionic Brownian Motion

Fermionic Brownian motion is obtained by defining a measure (in a generalised sense) on the space of paths in $\mathbb{R}_S^{0,n}$. First, an abstract "Grassmann measure" on the space $(\mathbb{R}_S^{0,n})^A$ of functions $\gamma : A \to \mathbb{R}_S^{0,n}$ is defined using the converse of the idea of the Kolmogorov construction, so that measures are

defined by their finite-dimensional marginal distributions. The ingredients for such a measure are thus a collection of functions $\{f_J\}$ corresponding to the finite subsets J of A; the function f_J has domain $(\mathbb{R}_S^{0,n})^J$ and satisfies the condition

$$\int d^n\theta_1 \dots d^n\theta_{|J|} f_J(\theta_1, \dots, \theta_{|J|}) = 1. \tag{11}$$

(Here $\theta_1, \dots, \theta_{|J|}$ are n-vectors.) The functions also satisfy a consistency condition: if $J = \{t_1, \dots, t_N\}$ and $J' = \{t_1 \dots, t_{N-1}\}$, then

$$\int d^n\theta_N f_J(\theta_1, \dots, \theta_N) = f_{J'}(\theta^1, \dots, \theta_{N-1}). \tag{12}$$

A Grassmann random variable is then defined to be any function on $(\mathbb{R}_S^{0,n})^A$ which can be integrated, possibly by some limiting process. When J is a finite subset of A and g is a function on $(\mathbb{R}_S^{0,n})^J$, then g is certainly a Grassmann random variable, with expectation defined to be

$$\mathbb{E}(g) = \int d^n\theta_1 \dots d^n\theta_{|J|} f_J(\theta_1, \dots, \theta_{|J|}) g(\theta_1, \dots, \theta_{|J|}). \tag{13}$$

Further details of the concept of Grassmann measure, and the related concept of Grassmann stochastic process may be found in [10].

Fermionic Brownian motion is an example of a Grassmann stochastic process; it is derived from Grassmann Wiener measure, which is a measure on $(\mathbb{R}_S^{0,n})^{(0,\infty)}$ with finite distributions

$$f_J(\theta_1, \rho_1, \dots, \theta_N, \rho_N) = \exp -i[\sum_{r=1}^{N} \rho_r.(\theta_r - \theta_{r-1})] \tag{14}$$

when $J = \{t_1, \dots, t_N\}$ with $0 < t_1, \dots, t_N \leq t$ (and $\theta_0 = 0$). (The only t_r-dependence in the measure is in the time ordering.) The corresponding $2n$-dimensional stochastic process (θ_t, ρ_t) will be referred to as Fermionic Brownian motion.

This measure is essentially composed of δ-functions, corresponding to the fact that the free Fermionic Hamiltonian is zero. Although in many ways this measure resembles conventional Wiener measure (which is based on the heat kernel of the free Bosonic Hamiltonian (or flat Laplacian) $-\frac{1}{2}\partial_i\partial_i$), it differs from it in that the Fourier transform or phase space variables ρ are

included. This means that a Feynman-Kac formula for differential operators of all orders can be developed. Explicitly, if H is the operator

$$H = V(\psi) = \sum_{\mu \in M_n} V_\mu \psi^\mu \tag{15}$$

(where, as in equation (4), $\psi^j = \theta^j + \frac{\partial}{\partial \theta^j}$), then

$$\exp(-Ht)\, f(\theta) = \mathbb{E}\Big[\exp -\Big(\int_0^t V(\omega(s))ds\Big) f(\theta + \theta(t))\Big] \tag{16}$$

where $\omega(t) = \theta(t) + i\rho(t)$, as is proved in [10].

3 Superspace Brownian motion

By mutiplying together finite-dimensional marginal distributions, Fermionic Wiener measure may be combined with conventional Wiener measure to give a notion of Brownian paths in superspace. These Brownian paths can then be used to develop Feynman-Kac formulae for a number of diffusion operators, as will be seen in later sections. The first step is to incorporate stochastic calculus into the framework of superspace Brownian paths.

There is no useful analogue of the Itô integral along Fermionic Brownian paths, because they are too irregular. However, since the motivation for considering these paths is to study diffusions, and the measure (incorporating the phase space variables ρ) is sufficient to handle differential operators of all orders, this does not matter. Also, Fermionic Brownian motion can be combined with conventional stochastic calculus, so that (if b_t denotes Brownian motion on $\mathbb{R}^m$) integrals of the form

$$Z_t - Z_0 = \int_0^t A_s\, ds + \int_0^t \sum_{a=1}^m C_{sa}\, db_s^a \tag{17}$$

can be constructed when A_s, C_s^a are suitably regular adapted stochastic processes on super Wiener space, and db_s are conventional Brownian increments. (Full details may be found in [11].)

Sufficiently well-behaved functions $G(Z_t^1, \dots, Z_t^p)$ of such integrals satisfy the Itô formula

$$\mathbb{E}\Big(G(Z_t^1, \dots, Z_t^p)\Big) - G(Z_0^1, \dots, Z_0^p)$$

$$= \mathbb{E}\Big(\int_0^t \sum_{j=1}^p A_s^j \partial_j G(Z_s^1, \ldots, Z_s^p) + \tfrac{1}{2} \sum_{a=1}^m \sum_{j=1}^p \sum_{k=1}^p C_{sa}^j C_{sa}^k \partial_k \partial_j G(Z_s^1, \ldots, Z_s^p) ds \tag{18}$$

where the partial derivative ∂_j has the same Grassmann parity as Z_t^j. This Itô formula can be used to give Feynman-Kac formulae for diffusions with drift in the usual way.

4 Stochastic differential equations for superspace Brownian paths

So far, it has been shown how Fermionic Brownian motion allows one to construct Feynman-Kac formulae for differential operators on functions on $\mathbb{R}_S^{m,n}$ which are second order in the even variables, with second order part simply the flat Laplacian $-\frac{1}{2}\sum_{i=1}^m \partial_i\partial_i$. In order to extend this approach to study operators in curved space, as is necessary both for geometrical applications and for the study of Fermions in a gravitational background, stochastic differential equations are required. As is shown in [11], stochastic differential equations of the form

$$\begin{aligned} dZ_s^j &= \sum_{a=1}^m A_a^j(Z_s, \theta_s, \rho_s, s) db_s^a + B^j(Z_s, \theta_s, \rho_s, s) ds \\ Z_0^j &= Z^j, \qquad j = 1, \ldots, p \end{aligned} \tag{19}$$

have unique solutions provided that the functions A_a^j and B^j are suitably regular.

The relevance of stochastic differential equations to this article is that they allow one to extend the scope of these stochastic methods for studying diffusions; as the following example shows, a wide class of second-order elliptic operators can be studied. (This example is a standard example from conventional stochastic calculus [6], included for those who are unfamiliar with this technique.)

Suppose that x_t^i satisfies the stochastic differential equation

$$\begin{aligned} dx_t^i &= \sum_{a=1}^m [V_a^i(x_t) db_t^a + \tfrac{1}{2} V_a^j(x_t)\, \partial_j V_a^i(x_t) dt] \\ x_0^i &= x^i \qquad i = 1, \ldots, p. \end{aligned} \tag{20}$$

It will now be shown that

$$\mathbb{E}[f(x_t)] = \exp(-Ht)f(x) \tag{21}$$

where H is the differential operator

$$H = -\tfrac{1}{2}\sum_{a=1}^{m} V^a V^a, \tag{22}$$

with $V_a = \sum_{i=1}^{p} V_a^i \partial_i$. This result is proved by applying the Itô formula to $f(x_t)$, which gives

$$\begin{aligned} f(x_t) - f(x) \quad &= \quad \int_0^t \sum_{a=1}^{m} V_a^i(x_s)\partial_i f(x_s) db_s^a + \int_0^t \tfrac{1}{2}\sum_{a=1}^{m} (V_a^i V_a^j \partial_i \partial_j) f(x_s) ds \\ &\quad + \int_0^t \tfrac{1}{2}\sum_{a=1}^{m}\sum_{i=1}^{p} \partial_j V_a^i V_a^j \partial_i f(x_s) ds. \end{aligned} \tag{23}$$

Thus, taking expectations of both sides,

$$\mathbb{E}[f(x_t)] - f(x) = \int_0^t \mathbb{E}[\tfrac{1}{2}\sum_{a=1}^{m} V^a V^a f(x_s)\,]ds \tag{24}$$

so that, if the operator U_t is defined by

$$U_t(g)(x) = \mathbb{E}[g(x_t)], \tag{25}$$

then we have shown that

$$U_t f(x) - f(x) = \int_0^t -U_s H(f)(x) ds \tag{26}$$

from which may be deduced that $U_t = \exp(-Ht)$ as required.

5 Brownian motion on supermanifolds

The technique of the previous section gives a Feynman-Kac formula for the operator $H = \sum_{a=1}^{m} V^a V^a$. In curved space the Laplacian is closely related to an operator of this form, and thus, following Elworthy [4] and Ikeda and Watanabe [6], the solutions to carefully constructed stochastic differential

equations can be used to study the heat kernels of the various Laplacians which occur on Riemannian manifolds. This approach can be extended to supermanifolds, as will now be described, leading to Feynman-Kac formulae for further Hamiltonians.

The appropriate supermanifold (for both geometric and physical applications) is constructed from a classical Riemannian manifold (M, g) of dimension m together with a smooth n-dimensional complex vector bundle E over M. The supermanifold $S(M, E)$ is of dimension $(m, m+n)$; it can be specified by stating the transition functions on overlapping coordinate patches, the actual supermanifold then being realised by a patching construction [9]. Suppose that $\{(U_\alpha, \phi_\alpha)\}$ is an atlas of charts on M (with each U_α also a trivialisation neighbourhood for E) and that $g_{\alpha\beta} : U_\alpha \cap U_\beta \to Gl(n, \mathbb{C})$ are the transition functions for the bundle E. Then the supermanifold is defined by taking m even coordinates $x^i_\alpha, i = 1, \ldots m$, m odd coordinates $\theta^i_\alpha, i = 1, \ldots m$ and n further odd coordinates $\eta^r_\alpha, r = 1, \ldots, n$, with transition functions

$$\begin{aligned} x^i_\alpha &= \phi^i_\alpha \circ \phi^{-1}_\beta(x_\beta) \\ \theta^i_\alpha &= \sum_{j=1}^{m} \frac{\partial(\phi^i_\alpha \circ \phi^{-1}_\beta)}{\partial x^j}(x_\beta)\theta^j_\beta \\ \eta^r_\alpha &= \sum_{s=1}^{n} {g_{\alpha\beta}}^r{}_s(\phi^{-1}_\beta(x_\beta))\eta^s_\beta. \end{aligned} \tag{27}$$

Equipped with this supermanifold it can be seen that E-valued forms on M (which locally can be expresses in the form

$$f(x) = \sum_{\mu \in M_n} \sum_{r=1}^{n} f_{\mu\, r}(x) dx^\mu e^r \tag{28}$$

where (e^r) is the local trivialisation basis of E) correspond to smooth functions on the supermanifold $S(M, E)$ which are linear in the η, these functions having the local form

$$f(x, \theta, \eta) = \sum_{\mu \in M_n} \sum_{r=1}^{n} f_{\mu\, r}(x) \theta^\mu \eta^r. \tag{29}$$

Under this correspondence the Laplace-Beltrami L operator on twisted forms becomes simply a differential operator, and so superspace stochastic calculus can be applied to study the corresponding heat kerenel. Specifically [12]

$$
\begin{aligned}
L &= \tfrac{1}{2}(d+\delta)^2 \\
&= -\tfrac{1}{2}\Big[\sum_{a=1}^{m} W_a W_a + \tfrac{1}{4}[\psi^i,\psi^j] F_{ij\,r}{}^{s}\,\eta^r \frac{\partial}{\partial \eta^s} \\
&\qquad + R_i^j(x)\theta^i \frac{\partial}{\partial \theta^j} - \tfrac{1}{2} R_{ki}{}^{j\ell}\theta^i\theta^k \frac{\partial}{\partial \theta^j}\frac{\partial}{\partial \theta^\ell}\Big]
\end{aligned}
\tag{30}
$$

where the summation convention is used and

$$
W_a = e_a^i \frac{\partial}{\partial x^i} - e_a^j e_b^k \Gamma^i_{jk} \frac{\partial}{\partial e_b^i} - e_a^j \theta^k \Gamma^i_{jk} \frac{\partial}{\partial \theta^i} - e_a^j \eta^r A^s_{j\,r} \frac{\partial}{\partial \eta^s}. \tag{31}
$$

(Here A and F are respectively the connection and curvature on E, and the supermanifold $S(M,E)$ is extended to include $O(M)$, the bundle of orthonormal frames; on $O(M)$ the local coordinates are $x^i, e_a^i, a, i = 1, \ldots, m$, corresponding to an orthonormal basis (e_a) of the tangent space, expanded in the basis of coordinate derivatives as $e_a = e_a^i \frac{\partial}{\partial x^i}$). The space of functions on which $W_a W_a$ acts is the space of functions on this extended supermanifold which are independent of the e_a^i.

Now suppose that $(x_t^i, \theta_t^i, \rho_t^i, \eta_t^i, e_{a\,t}^i)$ are solutions to the stochastic differential equation corresponding to W_a [12]. Then, arguing as in section 4, it can be shown that

$$
\exp(-Lt) g(x,\theta,\eta) = \mathbb{E}\Big[g(x_t,\theta_t,\eta_t)\Big]. \tag{32}
$$

6 Application to the index theorem

In general the stochastic differential equations of the previous section cannot be solved in closed form; however they can be applied to give estimates, particularly as t tends to zero. In addition to the usual estimate $b_t \sim \sqrt{t}$, one has $\theta_t \sim 1$ and $\eta_t \sim 1$.

One example of the application of these methods is the so-called supersymmetric proof of the Atiyah-Singer index theorem. Using the formula of McKean and Singer [8], and considering the case of the twisted Hirzebruch complex, this amounts to evaluating the 'supertrace' of $\exp(-Lt)$ (that is,

the trace of $\tau \exp(-Lt)$ where τ is an involution on forms) in the limit as t tends to zero. Using the identification established above of twisted forms with functions on the supermanifold $S(M, E)$, the required supertrace can be expressed as an integral of the kernel of $\exp(-Lt)$. In the original supersymmetric proofs of the index theorem of Alvarez-Gaumé [1] and of Friedan and Windy [5] somewhat heuristic arguments were used to show that when evaluating this supertrace the operator L can be replaced by a simpler operator whose kernel is well known. The stochastic approach described here allows this step to be made rigorous [12].

Asymptotic expressions for heat kerenels are also important in quantum gravity [3]. The stochastic calculus in this paper should make it possible to extend this approach to gravitational theories involving Fermions.

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